Domain Decomposition Methods for Time-Harmonic Electromagnetic Waves with High Order Whitney Forms

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Classically, domain decomposition methods (DDM) for time-harmonic electromagnetic wave propagation problems make use of the standard, low order, Nédélec basis functions. This paper analyses the convergence rate of DDM when higher order finite elements are used for both volume and interface discretizations, in particular when different orders are used in the volume and on the interfaces.

***Index Terms*—Computational electromagnetics, Wave propagation, Finite element analysis, Domain decomposition methods, High performance computing.**

I. Introduction

There is a growing consensus that state of the art finite element technology requires, and will continue to require, too extensive computational resources to provide the necessary resolution for complex high-frequency electromagnetic simulations. This leads us to consider methods with a higher order of grid convergence than the classical second order provided by most industrial grade codes.

Moreover, the direct application of the finite element method (FEM) on these high-frequency problems leads to very large, complex and possibly indefinite linear systems. Unfortunately, direct sparse solvers do not scale well for solving such large systems, and Krylov subspace iterative solvers can exhibit slow convergence, or even diverge [1]. Domain decomposition methods (DDM) provide an elegant alternative, iterating between subproblems of smaller sizes, amenable to sparse direct solvers [2].

In this paper we investigate the use of high order Whitney forms for the discretization of the subproblems as well as the interface conditions between the subdomains, and experiment with varying independently the discretization orders used in the volume and on the interfaces.

II. Problem Definition

Let us start by considering the time-harmonic propagation of an electrical wave $\mathbf{e}$ in an open waveguide Ω with metallic boundaries Γ^0. A source signal $\mathbf{e}^s$ is imposed on Γ^s. In order to solve this problem with the FEM, the infinite domain is truncated by a fictitious boundary Γ^∞, on which a Silver-Müller radiation condition is used. This leads to the following problem:

$$\left\{\begin{array}{rcll}\mathbf{curl}\,\mathbf{curl}\,\mathbf{e}-k^2\mathbf{e} & = & \mathbf{0} & \text{on } \Omega,\\ \gamma_{\mathrm{T}}(\mathbf{e}) & = & \mathbf{0} & \text{on } \Gamma^0,\\ \gamma_{\mathrm{T}}(\mathbf{e}) & = & \mathbf{e}^s & \text{on } \Gamma^s,\\ \gamma_{\mathrm{T}}(\mathbf{e})-\dfrac{\jmath}{k}\gamma_{\mathrm{t}}(\mathbf{curl}\,\mathbf{e}) & = & \mathbf{0} & \text{on } \Gamma^\infty,\end{array}\right. \tag{1}$$

where k is the wavenumber, $\jmath$ the imaginary unit, and the tangential trace and tangential component trace operators are given by $\gamma_{\mathrm{T}}(\mathbf{v}):\mathbf{v}\mapsto\mathbf{n}\times\mathbf{v}\times\mathbf{n}$ and $\gamma_{\mathrm{t}}(\mathbf{v}):\mathbf{v}\mapsto\mathbf{n}\times\mathbf{v}$, with $\mathbf{n}$ as the unit vector outwardly oriented normal to Ω.

III. Non-Overlapping Additive Schwarz Domain Decomposition Methods

Let us now review the construction of a non-overlapping additive Schwarz domain decomposition method for the propagation problem (1).

We start by splitting the domain Ω into non-overlapping subdomains Ω_i, with $i\in\{1,\dots,N_{dom}\}$. On a given subdomain Ω_i, the interface with subdomain Ω_j is denoted by Σ_{ij}. Conversely, on subdomain Ω_j, the interface with subdomain Ω_j is written Σ_{ji}. The electric filed on Ω_i is denoted by $\mathbf{e}_i$.

It can be shown [2] that the solution $\mathbf{e}$ of (1), on the whole domain Ω, can be computed by the following iterative scheme (indexed by p):

$$\left\{\begin{array}{rcll}\mathbf{curl}\,\mathbf{curl}\,\mathbf{e}_i^p-k^2\mathbf{e}_i^p & = & \mathbf{0} & \text{on } \Omega_i,\\ \gamma_{\mathrm{T}}(\mathbf{e}_i^p) & = & \mathbf{0} & \text{on } \Gamma_i^0,\\ \gamma_{\mathrm{T}}(\mathbf{e}_i^p) & = & \mathbf{e}^s & \text{on } \Gamma_i^s,\\ \gamma_{\mathrm{T}}(\mathbf{e}_i^p)-{}^{\jmath}/_{k}\,\gamma_{\mathrm{t}}(\mathbf{curl}\,\mathbf{e}_i^p) & = & \mathbf{0} & \text{on } \Gamma_i^\infty,\\ \mathcal{S}\left[\gamma_{\mathrm{T}}(\mathbf{e}_i^p)\right]-{}^{\jmath}/_{k}\,\gamma_{\mathrm{t}}(\mathbf{curl}\,e_i^p) & = & \mathbf{g}_{ij}^{p-1} & \text{on } \Sigma_{ij},\end{array}\right. \tag{2}$$

with

$$\mathbf{g}_{ij}^p=-\mathbf{g}_{ji}^{p-1}+2\,\mathcal{S}\left[\gamma_{\mathrm{T}}(\mathbf{e}_j^p)\right]\qquad\text{on } \Sigma_{ij}. \tag{3}$$

The quantity $\mathbf{g}_{ij}^p$ represents the coupling of Ω_i with Ω_j, and the operator $\mathcal{S}$ is a well chosen boundary transmission condition.

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A short presentation of optimized boundary conditions can be found in [3].

It is worth noticing [3] that solving iteratively (2) and (3) can be rewritten as the application of the iteration operator $\mathcal{A}$:

$$\mathbf{g}^p = \mathcal{A}\,\mathbf{g}^{p-1} + \mathbf{b},$$

where $\mathbf{g}^p$ is the concatenation of the $\mathbf{g}_{ij}^p$ for $1 \leq i, j \leq N_{dom}$, and $\mathbf{b}$ contains the contribution of the source electric field. Thus (2) and (3) can be solved using a Krylov solver applied to:

$$(\mathcal{I} - \mathcal{A})\mathbf{g} = \mathbf{b}, \tag{4}$$

where $\mathcal{I}$ is the identity operator. The set of subproblems in (2) can be solved independently and are of relatively small size, since they are defined on small subdomains. This property allows us to use (sparse) direct solvers.

IV. High Order Finite Element Discretization

Classically, DDM implementations make use of the standard Nédélec basis functions [4]. In this work, we propose to analyse the behaviour of the DDM when higher order bases are used, which are paramount to the accurate solution of high-frequency propagation problems thanks to their improved dispersion properties [5].

Using high order discretization may however lead to two significant drawbacks: a significant increase in the assembly time of the FEM matrix, and an increase in the iteration count of the DDM. The former can be addressed by newly proposed efficient and parallel high order assemblers [6]. The latter is currently an open problem, for which preliminary numerical tests are reported hereafter.

V. Numerical Experiments

For the numerical experiments, we consider the propagation problem defined in (1). The test cases will consist in varying the FEM discretization order of both equations (2) and (3). A reference solution is provided by an order 4 FEM solution computed without the DDM. The higher order Whitney forms are those proposed in [7].

The DDM transmission condition $\mathcal{S}$ used is simply the identity operator $\mathcal{I}$, corresponding to a Silver-Müller condition between the subdomains. Tests with other high order transmission conditions [8], [9] will be analyzed in the full paper.

Figure 1 summarizes the convergence rate of the GMRES for different FEM discretization orders of (2) and (3). The following notation is used to distinguish the possible orders: $\{v, s\}$, where v is the order used for (2) and s the order for (3).

Two behaviours are observed. First, for the dotted lines, the same FEM order was used. It can be directly seen that the highest the discretization order is, the slower the GMRES converges. This phenomenon might be explained by the discretization being able to represent faster parasitic oscillations. On the other hand, using a lower discretization order for the interface condition leads to a faster convergence of the DDM. This could be explained by the damping of parasitic oscillations through the fictitious DDM interfaces, introduced by the projection of $\mathbf{e}_i^p$ in (3) onto a smaller FEM subspace.

Table I summarizes the L_2 errors between the DDM and the FEM reference solution. Even for low order FEM discretization of (3), the L_2 error stays in acceptable ranges. However, the projection of $\mathbf{e}_i^p$ onto a smaller subspace may introduce numerical dissipation. This behaviour as well the overall hp convergence of the method will be analysed in the full paper.

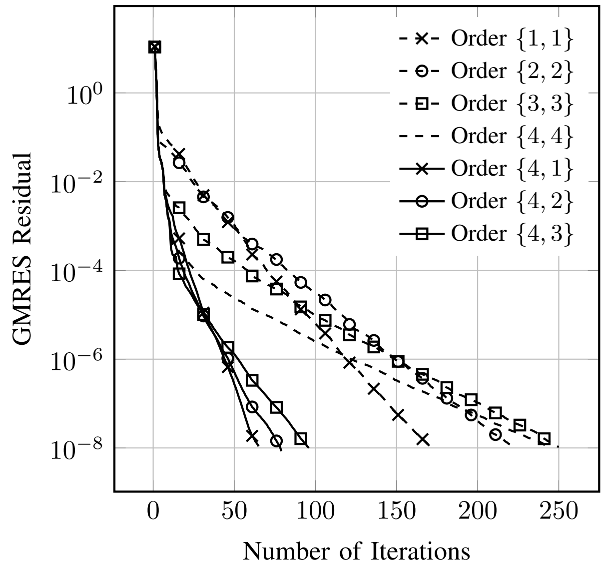

Fig. 1. Convergence rate of the GMRES for different discretization orders.

TABLE I
L_2 ERROR WITH RESPECT TO AN ORDER 4 FEM REFERENCE SOLUTION.

Order	L_2 Error
Order $\{4,1\}$	$1.3424\ 10^{-3}$
Order $\{4,2\}$	$5.7793\ 10^{-5}$
Order $\{4,3\}$	$8.5603\ 10^{-7}$

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